

THE REPRESENTATION ZETA FUNCTION OF A FAB COMPACT p -ADIC LIE GROUP VANISHES AT -2

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ABSTRACT. Let G be a compact p -adic Lie group and suppose that G is FAB, i.e., that $H/[H, H]$ is finite for every open subgroup H of G . The representation zeta function $\zeta_G(s) = \sum_{\chi \in \text{Irr}(G)} \chi(1)^{-s}$ encodes the distribution of continuous irreducible complex characters of G . For $p \geq 3$ it is known that $\zeta_G(s)$ defines a meromorphic function on \mathbb{C} .

Wedderburn's structure theorem for semisimple algebras implies that $\zeta_G(-2) = |G|$ for finite G . We complement this classic result by proving that $\zeta_G(-2) = 0$ for infinite G , assuming $p \geq 3$.

1. INTRODUCTION

Let G be a finitely generated profinite group, and let $\text{Irr}(G)$ denote the collection of all continuous irreducible complex characters of G . We observe that each $\chi \in \text{Irr}(G)$ has finite degree and for every positive integer $n \in \mathbb{N}$ we put $r_n(G) = |\{\chi \in \text{Irr}(G) \mid \chi(1) = n\}|$. From Jordan's theorem on finite linear groups in characteristic 0 (see [12, Theorem 9.2]) one deduces that $r_n(G)$ is finite for all $n \in \mathbb{N}$ if and only if G is FAB, i.e., if $H/[H, H]$ is finite for every open subgroup H of G .

Suppose that G is FAB. Then the arithmetic sequence $r_n(G)$, $n \in \mathbb{N}$, is encoded in the Dirichlet generating function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s} = \sum_{\chi \in \text{Irr}(G)} \chi(1)^{-s}$$

which is known as the *representation zeta function* of G . If the representation growth of G is polynomially bounded, i.e., if $\sum_{n=1}^N r_n(G) = O(N^d)$ for some constant d , then $\zeta_G(s)$ defines an analytic function on a non-empty right half-plane of \mathbb{C} . Under favourable circumstances, this function admits a meromorphic continuation, possibly to the entire complex plane \mathbb{C} .

In recent years representation growth and representation zeta functions have been investigated for various kinds of groups, including compact p -adic Lie groups; for instance, see [6, 9, 1, 2] or the short introductory survey [7]. An intriguing, but mostly unexplored aspect is the significance of special

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values of representation zeta functions. In particular, one may be curious about the locations of zeros and poles. While there is some theoretical understanding of the latter (see [2, Theorem B], and also compare [4] for the pole spectra of related zeta functions), almost nothing is known about the former.

In the present paper we establish that $\zeta_G(s)$ vanishes at $s = -2$ for every member G of a certain class of profinite groups, including all infinite FAb compact p -adic Lie groups for $p \geq 3$. Indeed, let G be a finitely generated profinite group which is FAb and virtually pro- p for some prime p . We say that G has *rational representation zeta function* (with respect to p), $\text{r.r.z.f.}_{(p)}$ for short, if there exist finitely many positive integers m_1, \dots, m_k and rational functions $R_1, \dots, R_k \in \mathbb{Q}(X)$ such that

$$(1) \quad \zeta_G(s) = \sum_{i=1}^k m_i^{-s} R_i(p^{-s}).$$

In [6], Jaikin-Zapirain proved that, for $p \geq 3$, every FAb compact p -adic Lie group has $\text{r.r.z.f.}_{(p)}$. It is conjectured that the result extends to 2-adic Lie groups; presently, it is known that every uniformly powerful pro-2 group has $\text{r.r.z.f.}_{(2)}$.

There is only a small number of FAb compact p -adic Lie groups G for which the representation zeta function $\zeta_G(s)$ has been computed explicitly; see [6, 1, 2]. By inspection of the formula given in [6, Theorem 7.5], Motoaki Kurokawa and Nobushige Kurokawa noticed that the representation zeta function of the p -adic Lie group $\text{SL}_2(\mathbb{Z}_p)$ has zeros at $s = -1$ and $s = -2$. The purpose of the present paper is to explain the zero at $s = -2$ which reflects a more general phenomenon.

Theorem 1. *Let G be a FAb profinite group which is infinite and virtually a pro- p group. If G has rational representation zeta function with respect to p then $\zeta_G(-2) = 0$.*

As indicated, using [6, Theorem 1.1] we derive the following corollary.

Corollary 2. *Let G be a FAb compact p -adic Lie group and suppose that $p \geq 3$. If G is infinite then $\zeta_G(-2) = 0$.*

Remark 3. Wedderburn's structure theorem for semisimple algebras implies that $\zeta_G(-2) = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G|$ for every finite group G . For an infinite profinite group G one can evaluate $\zeta_G(s)$ at $s = -2$ only if the function defined by the Dirichlet series has a suitable continuation.

Remark 4. The representation functions of compact open subgroups of semisimple p -adic Lie groups, such as $\text{SL}_n(\mathbb{Z}_p)$, occur naturally as factors in Euler products for the representation zeta functions of arithmetic lattices in semisimple groups, such $\Gamma = \text{SL}_n(\mathbb{Z}) \subseteq \text{SL}_n(\mathbb{R})$; see [9, Proposition 1.3]. However, since the Euler product formula is not valid for $s = -2$, one cannot use Corollary 2 directly to investigate potential properties of $\zeta_\Gamma(s)$ at $s = -2$. For instance, the inverse of the Riemann zeta function $\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n)n^{-s}$ satisfies $\zeta(s)^{-1} = \prod_p (1 - p^{-s})$ and $\zeta(0) = -2$.

In the next section we prove Theorem 1 and its corollary, by considering the p -adic limit of $\zeta_G(s)$ at $s = -2$. We also offer an alternative proof of

Corollary 2, which is closer to the character theoretic set-up in [6]. In the last section we provide further comments and highlight some open problems.

In conclusion, we remark that related questions regarding zeros and special values of Witten L -functions associated to real Lie groups, in particular to the groups $SU(2)$ and $SU(3)$, have been considered by Kurokawa and Ochiai [8] and also by Min [10].

2. THE PROOFS

Proof of Theorem 1. Let $m_1, \dots, m_k \in \mathbb{N}$ and $R_1, \dots, R_k \in \mathbb{Q}(X)$ such that the Dirichlet series $\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G)n^{-s} = \sum_{\chi \in \text{Irr}(G)} \chi(1)^{-s}$ satisfies (1). Then the degrees of the irreducible characters of G are of the form $m_i p^r$ with $i \in \{1, \dots, k\}$ and $r \geq 0$. In particular, for every positive integer j there are at most finitely many characters $\chi \in \text{Irr}(G)$ with $p^j \nmid \chi(1)$. Consequently, the series $\zeta_G(s)$ converges, with respect to the p -adic topology, at every negative integer $-e \in -\mathbb{N}$ to an element in the ring \mathbb{Z}_p of p -adic integers: we obtain a function

$$\zeta_G^{p\text{-adic}}: -\mathbb{N} \rightarrow \mathbb{Z}_p, \quad -e \mapsto \sum_{n=1}^{\infty} r_n(G)n^e = \sum_{\chi \in \text{Irr}(G)} \chi(1)^e.$$

For the last equality recall that in the p -adic topology every converging series converges unconditionally so that its summands can be re-arranged freely.

Equation (1) reflects more than the equality of two complex functions: by expansion of the right hand side we obtain a Dirichlet series whose coefficients must agree with the defining coefficients $r_n(G)$ of the zeta function on the left hand side. This implies that for every negative integer $-e$,

$$(2) \quad \zeta_G^{p\text{-adic}}(-e) = \sum_{i=1}^k m_i^e R_i(p^e) = \zeta_G(-e).$$

Consequently, it suffices to prove that $\zeta_G^{p\text{-adic}}(-2) = 0$.

Fix a positive integer j . As seen above, there are only finitely many characters $\chi \in \text{Irr}(G)$ such that $p^j \nmid \chi(1)$. We define

$$N_j = \bigcap_{\chi \in \text{Irr}(G), p^j \nmid \chi(1)} \ker \chi,$$

where each $\ker \chi$ coincides with the kernel of a representation affording χ . Then N_j is an open normal subgroup of G , and

$$\zeta_G^{p\text{-adic}}(-2) = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \sum_{\substack{\chi \in \text{Irr}(G) \text{ with} \\ N_j \subseteq \ker \chi}} \chi(1)^2 + \sum_{\substack{\chi \in \text{Irr}(G) \text{ with} \\ N_j \not\subseteq \ker \chi}} \chi(1)^2.$$

The first sum is equal to the order of the finite group G/N_j , while all the terms in the second sum are divisible by p^{2j} . Thus

$$(3) \quad \zeta_G^{p\text{-adic}}(-2) = |G : N_j| + p^{2j} a_j,$$

for some $a_j \in \mathbb{Z}_p$.

Since G is infinite and virtually a pro- p group, $|G : N_j| + p^{2j} a_j \rightarrow 0$ in the p -adic topology as $j \rightarrow \infty$. Thus (3) yields $\zeta_G^{p\text{-adic}}(-2) = 0$. \square

Next we give an alternative proof of Corollary 2, which is closer to the set-up in [6] and does not rely on p -adic limits.

Proposition 5. *Suppose that $p \geq 3$ and let N be a FAb uniformly powerful pro- p group. Then for every $m \geq 0$,*

$$\zeta_{N^{p^m}}(s) = |N : N^{p^m}| \zeta_N(s).$$

Proof. This is a consequence of the analysis in [2, Section 3] of a formula given in [6, Corollary 2.13]. \square

Proposition 6. *Suppose that $p \geq 3$ and let G be a FAb compact p -adic Lie group. Let H be an open subgroup of G . Then*

$$\zeta_G(-2) = |G : H| \zeta_H(-2).$$

Proof. Choose an open normal subgroup N of G which is 2-uniform (in the sense of [6, Section 2]) and contained in H . We show that

$$\zeta_G(-2) = |G : N| \zeta_N(-2).$$

The same reasoning yields $\zeta_H(-2) = |H : N| \zeta_N(-2)$, and combining the two equations proves the proposition.

We adapt the set-up in [6, Sections 5 and 6]. As in the proof of [6, Theorem 1.1], we decompose the representation zeta function of G as

$$\zeta_G(s) = \sum_{N \leq K \leq G} \sum_{\substack{\vartheta \in \text{Irr}(N) \\ \text{with } \text{St}_G(\vartheta) = K}} |G : K|^{-1-s} f_{(K,N,\vartheta)}(s) \cdot \vartheta(s)^{-1},$$

where for each character triple (K, N, ϑ) one defines

$$f_{(K,N,\vartheta)}(s) = \sum_{\chi \in \text{Irr}(K|\vartheta)} \left(\frac{\chi(1)}{\vartheta(1)} \right)^{-s}$$

summing over all $\chi \in \text{Irr}(K)$ such that ϑ is a component of $\text{red}_N^G(\chi)$. We observe that for each character triple (K, N, ϑ) ,

$$(4) \quad f_{(K,N,\vartheta)}(-2) = \sum_{\chi \in \text{Irr}(K|\vartheta)} \left(\frac{\chi(1)}{\vartheta(1)} \right)^2 = \frac{\text{red}_N^K(\text{ind}_N^K(\vartheta))(1)}{\vartheta(1)} = |K : N|.$$

It is proved in [6] that for each group K with $N \leq K \leq G$ the set $\text{Irr}(N)_K = \{\vartheta \in \text{Irr}(N) \mid \text{St}_G(\vartheta) = K\}$ can be partitioned into finitely many subsets $\text{Irr}(N)_{K,v}$, labelled by $v \in V_K$, such that

(i) for each $v \in V_K$ and $\vartheta \in \text{Irr}(N)_{K,v}$,

$$f_{(K,N,\vartheta)}(s) = f_v(s)$$

depends only on v and

(ii) for each $v \in V_K$,

$$g_v(s) = \sum_{\vartheta \in \text{Irr}(N)_{K,v}} \vartheta(s)^{-1}$$

is a rational function over \mathbb{Q} in p^{-s} .

The equations

$$\begin{aligned}\zeta_G(s) &= \sum_{N \leq K \leq G} \sum_{v \in V_K} |G : K|^{-1-s} f_v(s) g_v(s), \\ \zeta_N(s) &= \sum_{N \leq K \leq G} \sum_{v \in V_K} g_v(s)\end{aligned}$$

combined with (4) give

$$\begin{aligned}\zeta_G(-2) &= \sum_{N \leq K \leq G} \sum_{v \in V_K} |G : K| |K : N| g_v(-2) \\ &= |G : N| \zeta_N(-2).\end{aligned}$$

□

Second proof of Corollary 2. By Proposition 6 it is enough to prove the result for a uniformly powerful pro- p group N . By Propositions 5 and 6 we have

$$|N : N^p| \zeta_N(-2) = \zeta_{N^p}(-2) = |N : N^p|^{-1} \zeta_N(-2).$$

Since $|N : N^p| > 1$, this implies $\zeta_N(-2) = 0$.

□

3. OPEN QUESTIONS

We highlight three questions which arise naturally from Theorem 1, Corollary 2 and their proofs.

In view of (2) we record the following problem.

Question 1. Let G be a FAb compact p -adic analytic group. What are the values of $\zeta_G(s)$ at other negative integers $s = -e$ and is there a suitable interpretation of these?

Of course, we would like to extend Corollary 2 to the prime $p = 2$. More generally, one can ask the following.

Question 2. Let G be a FAb profinite group and suppose that $\zeta_G(s)$ converges in some right half-plane $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}$. Suppose further that $\zeta_G(s)$ has a meromorphic continuation so that $\zeta_G(-2)$ is defined. Is it true that $\zeta_G(-2) = 0$?

For instance it would be natural to investigate this question for compact analytic groups over compact discrete valuation rings of positive characteristic, e.g., over $\mathbb{F}_p[[t]]$. The representation zeta functions of such groups are still rather poorly understood. In particular, no analogue of Proposition 5 is known. However, a direct computation in [6] shows that, for $p \geq 3$, the group $\operatorname{SL}_2(\mathbb{F}_p[[t]])$ has the same representation zeta function as the p -adic analytic group $\operatorname{SL}_2(\mathbb{Z}_p)$.

The last question is inspired by Brauer's Problem 1, which asks: what are the possible degree patterns for irreducible characters of finite groups; see [3, 5, 11]. Given a profinite group G with r.r.z.f. $_{(p)}$, the completed group algebra $\mathbb{C}[[G]] = \varprojlim_{N \leq G} \mathbb{C}[G/N]$, formed with respect to the directed set of normal open subgroups of G , determines the representation zeta function of G and, conversely, $\zeta_G(s)$ determines $\mathbb{C}[[G]]$. Furthermore, if G is a pro- p group, then $\zeta_G(s)$ is a rational function over \mathbb{Q} in p^{-s} . The following can be regarded as an extension of Brauer's Problem 1 to FAb pro- p groups.

Question 3. Which rational functions $R(p^{-s})$ over \mathbb{Q} in p^{-s} are representation zeta functions of infinite FAb pro- p groups with r.r.z.f._(p)?

Theorem 1 provides a first necessary criterion: $R(p^2) = 0$.

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